Memoir on Seminvariants.

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I continue the discussion of the asyzygetic seminvariants which was commenced in the *American Journal of Mathematics*, Vol. VI, No. 2; reference is made also to two papers on the same subject by Professor Cayley and to one by myself in Vol. VII, No. 1 of the same Journal.

I will, for the present, consider the seminvariants of the quantic with derived coefficients

$$ax^n - nbx^{n-1}y + n(n-1)cx^{n-2}y^2 - n(n-1)(n-2)dx^{n-3}y^3 + \ldots$$

which are identical with the non-unitary symmetric functions of the equation

$$x^n - bx^{n-1} + cx^{n-2} - dx^{n-3} + \ldots = 0,$$

n being infinite, and which therefore satisfy the partial differential equation

$$d_b = \frac{d}{db} + b\frac{d}{dc} + c\frac{d}{dd} + d\frac{d}{de} + \ldots = 0.$$

Sec. 1. *The Three Cardinal Laws.*

The first may be called the derivation law and may be enunciated as follows: "The derivative of a form with respect to any letter is obtained with changed sign by performing the operation d_b upon the derivative with respect to the next preceding letter."

For using suffixed letters for convenience and putting

$$d_\lambda = a_0\frac{d}{da_\lambda} + a_1\frac{d}{da_{\lambda+1}} + a_2\frac{d}{da_{\lambda+2}} + \ldots$$

we have $$\frac{d}{da_\lambda} = d_\lambda - H_1 d_{\lambda+1} + H_2 d_{\lambda+2} - H_3 d_{\lambda+3} + \ldots$$

wherein H_w represents the total symmetric function of weight w according to the usual notation; whence

$$d_1 \frac{d}{da_\lambda} = d_1 d_\lambda - H_1 d_1 d_{\lambda+1} + H_2 d_1 d_{\lambda+2} - H_3 d_1 d_{\lambda+3} + \ldots$$
$$- d_{\lambda+1} + H_1 d_{\lambda+2} - H_2 d_{\lambda+3} + \ldots$$

since
$$d_\lambda H_w = (-)^{\lambda+1} H_{w-\lambda};$$

also d_λ, $d_{\lambda+1}, \ldots$ by their operation produce seminvariants (*vide* Hammond, Proc. Lon. Math. Soc., Vol. XIV, pp. 119-129), therefore

$$d_1 \frac{d}{da_\lambda} = -\frac{d}{da_{\lambda+1}},$$

which establishes the theorem.

This includes the theorem that the derivative with respect to the highest letter is itself a seminvariant, first proved I believe by Professor Sylvester (*cf. American Journal of Mathematics*, Vol. V, No. 1, p. 82). What appears to be the second fundamental property is in its essence Professor Sylvester's; it may be stated as follows: "If any seminvariant be operated upon by substituting for each letter, the letter of weight higher by unity, the resulting terms are those of highest degree in some seminvariant of higher weight"; this, as well as the converse proposition, is absolutely true in the case of the quantic with derived coefficients, but only true as regards forms when the binomial quantic is under consideration; it follows, of course, from a mere inspection of the operator d_b, and is in fact identical with the converse processes of 'capitation' and decapitation (*cf. Cayley, American Journal of Mathematics*, Vol. VII, No. 1).

I have remarked elsewhere (Quart. Jour. of Math., Vol. XX, No. 80) that 'diminishing' and 'decapitation' are alike the performance of the operator D_θ on a form of degree θ (*cf.* Hammond, *ante*).

The third property, which appears to be absolutely fundamental, may be termed 'The Conjugate Law,' which was foreshadowed in the symmetrical seminvariant tables in Vol. VI, No. 2, *American Journal of Mathematics*. It is in its essence an extension and refinement of Hermite's Law of Reciprocity, showing clearly the source whence that great classical theorem springs; the theorem is: "To every seminvariant corresponds another, the partitions of whose terms are the Ferrers-conjugates of the partitions of its own terms."

This appears from the following consideration: suppose P and Q to be conjugate terms, having therefore each the same number of different parts in

their partitions; by operating upon P with the component $a_\lambda \dfrac{d}{da_{\lambda+1}}$ of the operator d_1 a certain term will be produced, which, so far as its literal part is concerned, is merely P with the index of $a_{\lambda+1}$ diminished by unity; call this term p; now operating on Q with the element $a_\mu \dfrac{d}{da_{\mu+1}}$, $\mu+1$ being the $(\lambda+1)^{\text{th}}$ part of the partition of Q in descending order, a term q will be obtained, which is the conjugate of p; consequently the performance of d_1 on Q produces terms which are the conjugates of those obtained by performing d_1 on P, and there is an exact one-to-one correspondence; *ex. gr.* take the conjugate terms $a_1a_2a_3a_5a_7$ and $a_1^2a_2^2a_3a_4a_5$; $\dfrac{d}{da_1}$ and $a_4\dfrac{d}{da_5}$ operating on these respectively produce conjugate terms; so also do $a_1\dfrac{d}{da_2}$ and $a_3\dfrac{d}{da_4}$, $a_2\dfrac{d}{da_3}$ and $a_2\dfrac{d}{da_3}$, $a_4\dfrac{d}{da_5}$ and $a_1\dfrac{d}{da_2}$, $a_6\dfrac{d}{da_7}$ and $\dfrac{d}{da_1}$; it follows then that in seeking a form $\Sigma A.P$ by means of the differential equation, if we obtained an equation

$$\alpha A+\beta B+\gamma C+\ldots=0\,,$$

we must, in seeking a form $\Sigma A_1.Q$, arrive at a similar equation

$$\alpha_1A_1+\beta_1B_1+\gamma_1C_1+\ldots=0\,,$$

and that therefore if the form $\Sigma A.P$ exist, so also must the conjugate form $\Sigma A_1.Q$.

Another proof presents itself as a result of considering together Arbogast's Method of Derivations, and a new rule of Professor Cayley's, which, in some cases, is simpler in its performance; his rule for forming the combinations of a given degree and weight in the letters $(a, b, c, d, \ldots)$ from those of the same degree and of a weight lower by unity is (1) to multiply the latter throughout by b, with the exception of those terms already of the proper degree, and (2) further, to raise the weight of the first letter by unity, whenever such letter occurs to the first power only; now this rule and Arbogast's rule of the last, and the last but one letter are conjugate to one another, and by means of a scheme initiated by Professor Cayley establishes the conjugate law above enunciated. An example having reference to the form whose leading and ending terms are ei and e^3 respectively will show how this comes about.

Form $ei + e^3$		Conjugate Form $b^4c^4 + d^4$	
ARBOGAST'S RULE.		CAYLEY'S RULE.	
di	ei	b^5c^3	b^4c^4
bci	bdi	b^6cd	b^5c^2d
eh	c^2i	b^3c^4	b^6d^2
bdh	fh	b^4c^2d	b^2c^5
c^2h	beh	b^5d^2	b^3c^3d
fg	cdh	bc^5	b^4cd^2
beg	g^2	b^2c^3d	c^6
cdg	bfg	b^3cd^2	bc^4d
bf^2	ceg	c^4d	$b^2c^2d^2$
cef	d^2g	bc^2d^2	b^3d^3
d^2f	cf^2	b^2d^3	c^3d^2
de^2	def	cd^3	bcd^3
	e^3		d^4

It is clear from the above example that the last-and-last-but-one terms fg, cef, de^2 which by Arbogast's rule are doubly operated upon, give rise, in the conjugate form, to terms in which the first letter has an index unity, and which therefore are doubly operated upon by Cayley's rule; in fact the two rules are completely conjugate and rigorously establish the conjugate law. We might of course apply Cayley's rule to the form $ei + e^3$, and Arbogast's to the conjugate with equally satisfactory results.

It is to be observed in regard to the definition of the conjugate law above given, that in any form every literal term is supposed to be present that is at once posterior to the leading term in counter order* and anterior to the ending term in alphabetical order; it may happen that certain of these terms, not being any of them a leading or an ending term, are affected with zero coefficients; but it does not thence follow that the conjugate terms are absent in the conjugate form.

* The alphabetical order of terms is that of Prof. Cayley's Seminvariant Tables in Vol. VII, No. 1 of this Journal. The counter order is got by taking the conjugates of these in reverse order. The former proceeds according to degree only, and the latter simply by weight of highest letter. These names are Professor Cayley's, and he fully explains them in a memoir shortly to appear, I believe, in this Journal.

Sec. 2. *The formation of an Asyzygetic Series of Seminvariants of a given weight.*

It will be convenient, although not necessary, to consider a particular weight; suppose the weight 11. If we wish to discover the simplest source having an ending term b^2d^3, we have to combine the form (53^2) with those whose symbols are posterior in counter order to it, in such a manner that a maximum number of the terms of highest extent in the letters may vanish, and thus it is manifest that an ending term, as having a partition the conjugate of a non-unitary partition, must of necessity be power-ending; that is, it must contain the highest letter involved in it to a power greater than unity; hence also it follows that a leading term must be a non-unitary one, for by the conjugate law it must be conjugate to a power-ending term.

It becomes a matter of importance to determine the best arrangement of the terms for the purpose of presenting to view an asyzygetic series, bearing in mind that it is desirable that the leading and ending terms shall indicate respectively the highest letter and the degree of the form. In the *American Journal of Mathematics*, Vol. VII, No. 1, Professor Cayley has arranged them in alphabetical order, which is one essentially of degree only, and possesses the advantage that the ending term of necessity indicates the degree correctly; the leading term, however, does not with certainty involve the most advanced letter in the form. The author's tables I to XII in the same journal do as a fact so far indicate correctly both of these, but in neither case necessarily; such an arrangement (after Durfee) is well adapted to showing the symmetry consequent on the conjugate law, but is objectionable, for the reasons above given, in any theoretical discussion of the forms; table XI differs from the remainder, and shows a modified arrangement; the first half of the terms were ranked according to weight of highest letter (the self-conjugate terms excluded), no attention being paid to degree; the self-conjugate terms then come, followed by the conjugates of the first half of the terms in reverse order. The last half therefore proceeded essentially as regards degree only, and it would appear *à priori* that with no other continuous arrangement of terms would there exist so high a probability of the leading and ending terms showing respectively the highest letter and the degree of the source. '*In medio tutissimus ibis.*' However, such a sequence of terms would in fact only be less imperfect than others of a continuous nature, for it is manifestly impossible for the terms to proceed both as regards weight of highest letter and degree.

It appears then to be imperative to abandon any continuous order, and as a consequence the following method has been adopted: after Professor Cayley I call the order according to weight of highest letter the counter order. Consider the weight n, and starting with the letter of weight n proceed with terms which follow both the alphabetical and counter-orders, the alphabetical order being dominant; that is to say, the order is an alphabetical one, with those terms omitted which do not as well obey the counter-order; the remaining terms are then taken, and starting with the first one which was omitted from the first series, a second series is formed which is in both orders; of the remaining terms a third series is formed, and so forth, until all the terms are exhausted.

As an example the arrangement of weight 11 in four series is exhibited.

SERIES 1.	SERIES 2.	SERIES 3.	SERIES 4.
l	b^2j	b^3i	b^4h
bk	bci	b^2ch	b^5g
cj	bdh	bc^2g	b^6f
di	c^2h	b^3cg	
eh	cdg	b^3df	
fg	b^2dg	b^2c^2f	
beg	b^2ef	b^4cf	
bf^2	$bcdf$	b^4de	
cef	c^3f	b^5ce	
d^2f	b^3e^2	b^7e	
de^2	b^2cde		
bce^2	bc^3e		
bd^2e	b^3c^2e		
c^2de	b^3cd^2		
cd^3	b^5d^2		
b^2d^3	b^4c^2d		
bc^2d^2	b^6cd		
c^4d	b^8d		
b^2c^3d			
bc^5			
b^3c^4			
b^5c^3			
b^7c^2			
b^9c			
b^{11}			

Each of these series is, taken by itself, perfect in arrangement, for each is in both alphabetical and counter-order; further, each is self-conjugate; that is to say, if a series contain m terms, the r^{th} and the $(m-r+1)^{\text{th}}$ terms from the top are conjugate to one another; this must of necessity be so, for the process is a conjugate one.

Being given the leading and ending terms of a seminvariant of weight 11, an inspection of these series indicates the only possible terms that can occur in it; for instance, consider the form, written for brevity, $fg + b^2d^3$; we have to take in each series those terms which at once coincide with or are posterior to fg in counter-order, and coincide with or are anterior to b^2d^3 in alphabetical order; in series 1, 2, 3, we take the terms from fg to b^2d^3, from cdg to bc^3e, from bc^2g to b^2c^2f respectively; or if we merely required this form by itself, we might start with fg and write down in succession the portions of the series above indicated.

The asyzygetic table for any weight formed on this principle consists of as many separate blocks or parts as there are series of terms; each block of necessity possesses the reversible symmetry of my original tables, and the leading and ending terms must be respectively at the head and at the foot of the corresponding column of some one of the blocks; these two indicator terms do not necessarily appear in the same block.

I annex the new tabulation of weight 8, in two portions, for the binomial quantic.

ASYZYGETIC SEMINVARIANTS

Weight 8.

		Col. $i+e^2$	Col. $cg+cd^2$	Col. $df+b^2d^2$	Col. e^2+c^4	Col. $ce^2+b^2c^3$	Col. $cd^2+b^4c^2$	Col. c^4+b^8
	i	$+1$						
	bh	-8						
	cg	$+28$	$+1$					
	df	-56	-3	$+3$				
	e^2	$+35$	$+2$	-2	$+1$			
	bde		-1	$+1$	-8			
	c^2e		-3	$+18$	$+6$	$+1$		
	cd^2		$+2$	-12		-1	$+1$	
Part 1.	b^2d^2			$+10$	$+16$	$+1$	-1	
	bc^2d				-24	$+2$	-6	
	c^4				$+9$	-1	$+4$	$+1$
	b^2c^3					$+1$	-7	$+4$
	b^4c^2						$+3$	$+6$
	b^6c							-4
	b^8							$+1$
	b^2g		-1					
	bcf		$+3$	-9				
	b^3f			$+6$				
Part 2.	b^2ce			-15		-2		
	b^4e					$+1$		
	b^3cd					-2	$+10$	
	b^5d						-4	

This table compares, I think, favorably with the preceding one in many ways; the only blank spaces that can occur must be consequent upon accidental zeros, so that the numbers are brought as close as possible together; should it be found necessary to tabulate weights higher than 12, the splitting up into blocks may be found of considerable advantage.

The minimum forms. By a minimum form I mean the form of lowest degree (*i. e.* the one possessing the earliest ending term in alphabetical order) that has a

given leading term; or conversely, given a particular ending term, the minimum form is that one whose leading term is the latest possible in counter-order. In this investigation, for a given weight, we can by simple observation of the known results in the case of lower weights at once narrow the inquiry by imposing limits as regards the degree and weight of highest letter of the ending and leading terms that can be associated with given leading and ending terms respectively; weight 13 will be taken as an example of this, the limits of the forms being determined by means of those in my tables I to XII, which are indubitably correct.

I arrange the leading or non-unitary partition terms in counter-order, that is, according to weight of highest letter only; and by reason of the conjugate law, as well as from considerations which will appear in the sequel, it is natural to place the ending or power-ending partition terms in alphabetical order. Placing these in two vertical columns, columns of limits are ranged to the left and right of these on the following principle.

The limits. The form $(n + *)$ is obviously of degree $\not< 3$; pass on to form $(cl + *)$; by the first fundamental law '*ante,*' the derivative with respect to the highest letter l must be a seminvariant and cannot therefore be other than $(c + b^2)$; thus $(cl + *)$ degree $\not< 3$.

Similarly $(dk + *)$ degree $\not< 4$

$(ej + *)$ degree $\not< 3$

since there exists the minimum form $(e + c^2)$ in Table IV.

And so forth a limit of degree is assigned for the ending term associated with each leading term. For the column of ending terms we can assign a limit to the weight of the highest letter in the associated leading term; for by the second law '*ante,*' when each letter of an ending term is withdrawn one place in alphabetical order, it must be the ending term of a form of lower weight, and the whole of the literal terms of highest degree must be derived from some form of lower weight by advancing each letter therein one place in alphabetical sequence; for example, from the ending term bcf^2, we obtain abe^2, and since Table IX displays a minimum form $(j + be^2)$, the minimum character of the highest-degree terms in the form ending bcf^2 must be $(b^3k + bcf^2)$; that is,

$$(* + bcf^2) \text{ letter } \not< k.$$

These two conjugate processes may be shortly expressed as follows

Leading	Ending
Decapitate	Diminish
Complete	Complete
Deg. $\not<$ (deg. ending + 1).	Letter $\not<$ (letter of leading + 1)

thus:	cg^2	$b^2c^3d^2$
	cg	b^3c^2
	$cg + cd^2$	$de + b^3c^2$
	Deg. $\nless 4$	Letter $\nless f$.

We thus have as under:

THE ALGORITHM $w = 13$.

Shown by	Degree	Lead term	End term	Letter	Shown by
	$\nless 3$	n	bg^2	$\nless n$	
$c + b^2$	$\nless 3$	cl	df^2	$\nless j$	$ci + ce^2$
$d + b^3$	$\nless 4$	dk	bcf^2	$\nless k$	$j + be^2$
$e + c^2$	$\nless 3$	ej	be^3	$\nless i$	$ch + d^3$
$c^2 + b^4$	$\nless 5$	c^2j	cde^2	$\nless h$	$dg + bcd^2$
$f + bc^2$	$\nless 4$	fi	b^3f^2	$\nless j$	$i + e^2$
$cd + b^5$	$\nless 6$	cdi	b^2de^2	$\nless h$	$cg + cd^2$
$\rho\theta - 2w \nless 0$	$\nless 4$	gh	bc^2e^2	$\nless g$	$df + b^2d^2$
$ce + c^3$	$\nless 4$	ceh	bd^4	$\nless g$	$\rho\theta - 2w \nless 0$
$d^2 + b^2c^2$	$\nless 5$	d^2h	c^2d^3	$\nless g$	$\rho\theta - 2w \nless 0$
$c^3 + b^6$	$\nless 7$	c^3h	b^3ce^2	$\nless i$	$h + bd^2$
$cf + bc^3$	$\nless 5$	cfg	b^2cd^3	$\nless g$	$cf + bc^3$
$de + b^3c^2$	$\nless 6$	deg	bc^3d^2	$\nless f$	$de + b^3c^2$
$c^2d + b^7$	$\nless 8$	c^2dg	b^5e^2	$\nless h$	$g + d^2$
$\rho\theta - 2w \nless 0$	$\nless 6$	df^2	b^4d^3	$\nless f$	$ce + c^3$
$\rho\theta - 2w \nless 0$	$\nless 6$	e^2f	$b^3c^2d^2$	$\nless e$	$d^2 + b^2c^2$
$c^2e + b^2c^3$	$\nless 6$	c^2ef	bc^6	$\nless e$	$\rho\theta - 2w \nless 0$
$cd^2 + b^4c^3$	$\nless 7$	cd^2f	b^5cd^2	$\nless g$	$f + bc^2$
$c^4 + b^8$	$\nless 9$	c^4f	b^3c^5	$\nless e$	$cd + b^5$
$cde + b^3c^3$	$\nless 7$	cde^2	b^7d^2	$\nless f$	$e + c^2$
$d^3 + b^5c^2$	$\nless 8$	d^3e	b^5c^4	$\nless d$	$c^2 + b^4$
$c^3d + b^9$	$\nless 10$	c^3de	b^7c^3	$\nless e$	$d + b^3$
$c^2d^2 + b^6c^2$	$\nless 9$	c^2d^3	b^9c^2	$\nless d$	$c + b^2$
	$\nless 13$	c^5d	b^{13}	$\nless d$	

It will be noted that in some cases closer limits are secured by the condition $\rho\theta - 2w \not< 0$ wherein $\rho =$ extent, $\theta =$ degree, $w =$ weight, and it does not appear whether such closer limits are absolutely necessary to the present scheme.

Now I suggest that, beginning from the top, the lead terms are to be paired off with the earliest end terms that satisfy the required conditions, as shown by the limits to the left and right of the columns, and that such an algorithm will give an existent, and furthermore, a minimum system of seminvariants.

I observe that the weights of the letters in the right-hand column of limits are the same as the numbers in the degree column in reverse order, and this is true universally from the conjugate nature of the process. For a like reason the joining lines present the same appearance when the table is inverted, and it is in the symmetry of the algorithm that the *à posteriori* probability of its truth lies. I proceed to its examination. Consider a weight w, supposed odd for convenience, and the cubic form

$$\phi_3 \equiv A_3 3^p 2^{\frac{1}{2}(w-3p)} + B_3 3^{p-2} 2^{\frac{1}{2}(w-3p+6)} + C_3 3^{p-4} 2^{\frac{1}{2}(w-3p+12)} + \dots$$

in which the form $3^p 2^{\frac{1}{2}(w-3p)}$ is to be combined with forms of lower degree so that a maximum number of terms of higher extent shall vanish; we shall thus arrive at a minimum form whose ending term possesses a partition the conjugate of the partition $3^p 2^{\frac{1}{2}(w-3p)}$, we have to determine the coefficients A_3, B_3, $C_3 \dots$ (suppose q_3 in number) that such may be the case, and we can only do so by finding $q_3 - 1$ independent linear relations between them.

We need only to consider the non-unitary terms in ϕ_3, since a form is completely given by its non-unitary portion, and we must express in succession the conditions that a_w, a_2a_{w-2}, $a_3a_{w-3} \dots$ may be absent from ϕ_3; hence we must have $\frac{d\varphi}{da_w} = 0$, *i. e.* $d_w\phi = 0$ or $[w]\phi = 0$ where the portions in [ ] refer to the symmetric functions of the equation

$$1 - y^{-1}D_1 + y^{-2}D_2 - y^{-3}D_3 + \dots = 0,$$

D_λ being as usual Mr. Hammond's operator.

In operating with $[w]$ on ϕ we are obviously only concerned with the cubic equation

$$1 + y^{-2}D_2 - y^{-3}D_3 = 0,$$

and we thus get one relation between A_3, B_3, C_3, ...; similarly operating with $[w - 2.2]$, $[w - 3.3]$, $[w - 4.4]$, $[w - 4.2^2]$, ... we obtain the relations that must be satisfied by the coefficients if the terms a_2a_{w-2}, a_3a_{w-3}, a_4a_{w-4}, $a_2^2a_{w-4}$, ... be not present.

The question that now arises is: How many of these operations $[w]$, $[w-2.2]\ldots$ are independent.

The Syzygies.

Consider now not the cubic forms but those of any degree θ; we have the equation of degree θ

$$x^\theta + a_2x^{\theta-2} - a_3x^{\theta-3} + \ldots \pm a_\theta = 0;$$

and its non-unitary symmetric functions of weight w $(w > \theta)$ whose partitions contain not more than θ parts; the linear relations connecting them are found by eliminating between them all the non-unitary terms which contain no part $> \theta$; the number of these syzygies will obviously be the excess of the number of non-unitary partitions of w having not more than θ parts over the number of non-unitary partitions having no part greater than θ; that is it will be

$$\begin{aligned}
&\text{Co. } a^\theta x^w \frac{1}{(1-a)(1-ax)\ldots(1-ax^w)} - \text{Co. } a^{\theta-1}x^{w-1}\frac{1}{(1-a)(1-ax)\ldots(1-ax^w)} \\
&\qquad - \text{Co. } x^w \frac{1}{(1-x^2)(1-x^3)\ldots(1-x^\theta)} \\
&= \text{Co. } x^w \frac{1}{(1-x)(1-x^2)\ldots(1-x^\theta)} - \text{Co. } x^{w-1}\frac{1}{(1-x)(1-x^2)\ldots(1-x^{\theta-1})} \\
&\qquad - \text{Co. } x^w \frac{1}{(1-x^2)(1-x^3)\ldots(1-x^\theta)} \\
&= \text{Co. } x^{w-\theta-1}\frac{1}{1-x.1-x^2.1-x^3\ldots1-x^\theta},
\end{aligned}$$

that is to say, their number is equal to the number of partitions of $w-\theta-1$ having no part $> \theta$.

Turning again to the cubic forms and the cubic equation, we can easily form the actual syzygies, for by the Newtonian theorem of the sums of powers

$$(w) + c\,(w-2) - d\,(w-3) = 0,$$

and

$$(3) - 3d = 0$$

$$(2) + 2c = 0,$$

whence a first syzygy

$$(w) - 3\,(w-2.2) - 2\,(w-3.3) = 0, \tag{1}$$

which, in the case of cubic forms, indicates a linear relation between the equations derived by means of the operators $[w]$, $[w-2.2]$, $[w-3.3]$; hence if a_w and a_2a_{w-2} be absent, so also must be a_3a_{w-3}, which is obviously consistent with what has gone before, since we know that a_3a_{w-3} cannot be the lead term of a cubic form.

Again, clearly $\qquad (w-4)\{(4) + c\,(2)\} = 0,$

or

$$(w) - 2\,(w-2.2) + (w-4.4) - 2\,(w-4.2^2) = 0; \tag{2}$$

the next syzygy, showing, what we know otherwise, that $a_2^2a_{w-4}$ cannot be the lead term of a cubic form.

This second syzygy is derived from the first by putting $w=4$ and then multiplying by $(w-4)$.

The next two are obtained by putting $w=5$ and $=w-2$ and multiplying by $(w-5)$ and (2) respectively; thus

$$(w)-5(w-2.2)-5(w-3.3)+(w-5.5)-5(w-5.3.2)=0 \qquad (3)$$

$$(w)-2(w-2.2)-2(w-3.3)-3(w-4.4)-6(w-4.2^2)$$
$$-2(w-5.5)-2(w-5.3.2)=0 \qquad (4)$$

which may be written

$$(w)-5(w-2.2)-5(w-3.3)+(w-5.5)-5(w-5.3.2)=0$$
$$(w)-5(w-4.4)-10(w-4.2^2)-4(w-5.5)=0,$$

indicating that $a_2a_3a_{w-5}$ and a_5a_{w-5} are impossible lead terms for cubic forms.

Again in (1) putting $w=6$ and $w-3$ and multiplying respectively by $(w-6)$ and (3) we obtain the syzygies showing that $a_2a_4a_{w-6}$ and $a_3^2a_{w-6}$ cannot be lead terms. Next

put in (1)	$w=7$	multiply by	$(w-7)$
	$w=5$	. . . . .	$(w-7)(2)$
	$w=4$	. . . . .	$(w-7)(3)$
	$w=w-4$	. . .	(4)

(N. B.—We do not also multiply the latter by (2^2) by reason of the relation $(4)-2(2^2)=0$); between these four syzygies we eliminate the term $(w-7.3.2^2)$ thus obtaining three significant syzygies showing the nullity of the terms a_7a_{w-7}, $a_2a_5a_{w-7}$, $a_3a_4a_{w-7}$.

This method is easily continued and forces on us the conclusion that cubic forms exist for each lead term a_0a_w, a_2a_{w-2}, a_4a_{w-4}, a_6a_{w-6}, ..., taken in succession until all the cubic forms are exhausted; those terms of this type which remain over must belong to higher forms. The algorithm is therefore proved so far as regards cubic forms

Passing on now to quartic forms I discuss the linear relations connecting the non-unitary symmetric functions of the equation

$$x^4+cx^2-dx+e=0$$

for a weight $w\,(>4)$.

These, as before, will indicate the impossible lead terms of quartic seminvariants.

We have

$$(w)+c(w-2)-d(w-3)+e(w-4)=0 \qquad (\alpha)$$
$$(4)+c(2)+4e=0$$
$$(3)-3d \qquad =0$$
$$(2)+2c \qquad =0$$

whence a first syzygy

$$(w)-18(w-2.2)-8(w-3.3)-9(w-4.4)-3(w-4.2^2)=0$$

or $a_2^2a_{w-4}$ is not a lead term.

Put now in (α) $w=5$ and multiply by $(w-5)$, whence the next syzygy

$$(w)-5(w-2.2)-5(w-3.3)+(w-5.5)-5(w-5.3.2)=0$$

or $a_2a_3a_{w-5}$ is not a lead term; a_5a_{w-5} may be and will be if there be sufficient quartic forms left after the other prior possible lead terms have been utilized.

As before we next have a batch of two syzygies obtained by putting in (α)

$$\begin{array}{lll} w=6 & \text{and multiply by} & (w-6) \\ w=w-2 & \text{and} \quad `` & (2); \end{array}$$

without working these out it is easy to see that they will indicate syzygies corresponding to the terms $a_3^2a_{w-6}$, $a_2^3a_{w-6}$.

Again in (α) put

$$\begin{array}{llll} w=7 & \text{and multiply by} & & (w-7) \\ w=5 & `` & `` & (w-7)(2) \\ w=w-3 & `` & `` & (3) \end{array}$$

thus obtaining syzygies which negative $a_2a_5a_{w-7}$, $a_3a_4a_{w-7}$, $a_2^2a_3a_{w-7}$. Also put

$$\begin{array}{llll} w=8 & \text{and multiply by} & & (w-8) \\ w=6 & `` & `` & (w-8)(2) \\ w=5 & `` & `` & (w-8)(3) \\ w=w-4 & `` & `` & (4) \end{array}$$

and syzygies will result which negative $a_3a_5a_{w-8}$, $a_4^2a_{w-8}$, $a_2^2a_4a_{w-8}$, $a_2a_3^2a_{w-8}$ as lead terms of quartic forms.

The syzygies are readily continued and material thus obtained for determining '*seriatim*' the lead terms, in counter order, which are associated with the end terms in alphabetical order.

No difficulty occurs in the treatment of the quintic and higher minimum seminvariant forms by the same method; the first syzygy in the case of forms of degree θ is obtained by the elimination of $c, d, e, \ldots, a_\theta$ between the equations

$$\begin{array}{ll} (w)+c(w-2)-d(w-3)+e(w-4)-\ldots+(-)^\theta a_\theta(w-\theta) & =0 \\ (\theta)+c(\theta-2)-d(\theta-3)+e(\theta-4)-\ldots+(-)^\theta\theta a_\theta & =0 \\ (\theta-1)+c(\theta-3)-d(\theta-4)+e(\theta-5)-\ldots+(-)^{\theta-1}\theta-1.a_{\theta-1} & =0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdot \\ (3)-3d & =0 \\ (2)+2c & =0 \end{array}$$

and the remaining syzygies in successive batches as before.

Cases occur in which one or more of the coefficients are arbitrary, and the form therefore is, strictly speaking, indeterminate; for example, take the form of weight 17,

$$\phi_4 \equiv A_4 (432^5) + B_4 (3^5 2) + C_4 (3^3 2^4) + D_4 (32^7);$$

the conditions that r and cp may be absent, give two relations between the coefficients A_4, B_4, C_4, D_4, and since the term 'en' can also be made to vanish with 'do' if $A_4 = 0$, it follows that the form $do + bch^2$ properly includes the form $en + fg^2$, and is therefore indeterminate; we can, however, agree to fix the form $do + bch^2$ by making r, cp and en vanish, thus forming three independent equations between the four coefficients; in general a form is fixed by setting apart the lead term and making a maximum number of the remaining non-unitary terms, in counter order, vanish.

The foregoing investigation so far agrees with the algorithm previously set forth, but is, however, insufficient to completely establish it; I do not at present see the way to effect this.

I have verified it as far as weight 17 inclusive.

SEC. 3. *The Calculation of Seminvariants.*

I indicate a method of calculating the covariant sources of the quantic

$$(1,\ 0,\ a_2,\ a_3,\ a_4,\ \ldots \between x,\ y)^n.$$

It is well known that if $\phi(a_2, a_3, a_4, \ldots)$ be any such source, $\phi(A_2, A_3, A_4, \ldots)$ will be the corresponding form for the quantic

$$(1,\ a_1,\ a_2,\ a_3,\ a_4,\ \ldots \between x,\ y)^n,$$

where

$$A_r = a_r - ra_1 a_{r-1} + \tfrac{1}{2} r(r-1) a_1^2 a_{r-2} - \ldots + (-)^r \tfrac{1}{2} r(r-1) a_1^{r-2} a_2 + (-)^{r+1} (r-1) a_1^r.$$

A_r being itself a source, ϕ may be any function whatever, or what is the same thing, ϕ being arbitrary, we have $\phi(a_2, a_3, a_4, \ldots)$ as the non-unitary part of a seminvariant: we can, by the proper determination of the arbitrary constants, reduce the degree of this seminvariant, viz.,

$$A_2 = a_2 - a_1^2$$

$$A_4 = a_4 - 4a_1 a_3 + 6a_1^2 a_2 - 3a_1^4,$$

$a_4 + ka_2^2$ is the non-unitary part of $a_4 - 4a_1 a_3 + 6a_1^2 a_2 - 3a_1^4 + k(a_2^2 - 2a_1^2 a_2 + a_1^4)$ a seminvariant in general of degree 4, but which for $k = 3$, becomes $a_4 - 4a_1 a_3 + 3a_2^2$ of degree 2.

Suppose $\phi = \Sigma L a_{\lambda_1} a_{\lambda_2} a_{\lambda_3} \ldots a_{\lambda_\kappa}$, a_{λ_1}, $a_{\lambda_2} \ldots$ being of course non-unitary, I proceed to find the relations that must exist between the L's, in order that the successive possible ending terms may vanish; these are power-ending terms, and

I will consider them arranged so that their conjugates are in counter-order. Thus

End term.	Conjugate.
a_1^w	a_w
$a_1^{w-4}a_2^2$	$a_{w-2}a_2$
$a_1^{w-6}a_2^3$	$a_{w-3}a_3$
$a_1^{w-8}a_2^4$	$a_{w-4}a_4$
$a_1^{w-6}a_3^2$	$a_{w-4}a_2^2$
$\vdots$	$\vdots$

What is done is in reality to establish a rule for the product of any number of terms of the series $A_2, A_3, \dots$ and then to apply it to a seminvariant considered as a sum of any number of such products.

In order that a_1^w may vanish we must have

$$\Sigma L(-)^{\lambda_1+1}(\lambda_1-1)(-)^{\lambda_2+1}(\lambda_2-1)\dots(-)^{\lambda_\kappa+1}(\lambda_\kappa-1)=0,$$

or say
$$\Sigma(-)^\kappa \Lambda L = 0, \tag{1}$$
wherein
$$\Lambda=(\lambda_1-1)(\lambda_2-1)\dots(\lambda_\kappa-1);$$
a relation that must be satisfied by the non-unitary portion of every seminvariant of degree $<$ weight.

The condition for the vanishing of the non-power-ending term $a_1^{w-2}a_2$ is similarly found to be
$$\Sigma(-)^\kappa\Lambda\Sigma\lambda_1 L=0,$$
which, since $\Sigma\lambda_1 = w$, is implied in the former, and so in every case it will be found, what we know otherwise, that we need only to consider the power-ending terms.

Let $[1^j]$ denote the sum of the products j together of the integers $\lambda_1, \lambda_2, \dots \lambda_\kappa$, then for the vanishing of

$$a_1^{w-4}a_2^2 \text{ relation is } \Sigma(-)^\kappa\Lambda[1^2]L=0 \tag{2}$$
$$a_1^{w-6}a_2^3 \quad \text{``} \quad \Sigma(-)^\kappa\Lambda[1^3]L=0 \tag{3}$$
$$a_1^{w-8}a_2^4 \quad \text{``} \quad \Sigma(-)^\kappa\Lambda[1^4]L=0 \tag{4}$$
$$\vdots \qquad\qquad \vdots$$
$$a_1^{w-2j}a_2^j \quad \text{``} \quad \Sigma(-)^\kappa\Lambda[1^j]L=0.$$

In the case $a_1^{w-6}a_3^2$ we have $\Sigma(-)^\kappa\Lambda\Sigma\lambda_1\lambda_2(\lambda_1-2)(\lambda_2-2)L=0$, which may be reduced to a very simple form if we suppose relations (1), (2), (3), (4) to be already satisfied; for

$$\begin{aligned}\Sigma\lambda_1\lambda_2(\lambda_1-2)(\lambda_2-2) &= [1^2]^2-2[1][1^3]+2[1^4]-2[1][1^2]+6[1^3]+4[1^2],\\ &=[1^2]^2-2w[1^3]+2[1^4]-2w[1^2]+6[1^3]+4[1^2],\end{aligned}$$

or we have
$$a_1^{w-6}a_3^2 \text{ relation is } \Sigma(-)^\kappa\Lambda[1^2]^2L=0, \tag{5}$$

or this implies that the terms a_1^w, $a_1^{w-4}a_2^2$, $a_1^{w-6}a_2^3$, $a_1^{w-8}a_2^4$, being absent, so also must be $a_1^{w-6}a_3^2$.

So again for $a_1^{w-8}a_2a_3^2$, we have

$$\Sigma(-)^\kappa \Lambda \Sigma \lambda_1\lambda_2\lambda_3(\lambda_1-2)(\lambda_2-2)L=0,$$

and
$$\begin{aligned}\Sigma\lambda_1\lambda_2\lambda_3(\lambda_1-2)(\lambda_2-2) &= \Sigma\lambda_1\lambda_2(\lambda_1-2)(\lambda_2-2)(w-\lambda_1-\lambda_2)\\ &\equiv \Sigma\lambda_1\lambda_2(\lambda_1-2)(\lambda_2-2)(\lambda_1+\lambda_2) \text{ from (5)}\\ &\equiv [32]-2[31]-4[2^2]+4[21]\\ &\equiv [1^2][1^3] \text{ by (1), (2), (3), (4) and (5).}\end{aligned}$$

The condition that $a_1^{w-8}a_2a_3^2$ may vanish, as well as those before considered, is consequently $a_1^{w-8}a_2a_3^2 \qquad \Sigma(-)^\kappa\Lambda[1^2][1^3]L=0.$ (6)

From an examination of the method of obtaining the above relations the following general theorem is seen to be true.

Theorem.—"If $\Sigma La_{\lambda_1}a_{\lambda_2}\dots a_{\lambda_\kappa}$ be the non-unitary portion of a seminvariant whose developed expression contains no term, the conjugate of which is prior in counter-order to the non-unitary term $a_\alpha a_\beta^b a_\gamma^c a_\delta^d\dots$; then the condition that the term conjugate to $a_\alpha a_\beta^b a_\gamma^c a_\delta^d\dots$ may also be absent is

$$\Sigma(-)^\kappa\Lambda[1^\beta]^b[1^\gamma]^c[1^\delta]^d\dots L=0$$

wherein $\Lambda=(\lambda_1-1)(\lambda_2-1)\dots(\lambda_\kappa-1)$, and $[1^j]$ is the sum of the products j together of the integers $\lambda_1, \lambda_2, \dots \lambda_\kappa$."

This theorem obviously enables the calculation of any seminvariant from its non-unitary portion by the formation of a sufficient number of relations between its numerical coefficients, *ex. gr.* there is a quartic invariant containing the non-unitary terms a_2a_4, a_3^2, a_2^3, or say the portion in question is

$$Aa_2a_4+Ba_3^2+Ca_2^3.$$

Hence for

a_1^6 $(2-1)(4-1)A+(3-1)(3-1)B-(2-1)(2-1)(2-1)C=0$ or $3A+4B-C=0$
$a_1^2a_2^2$ 8 " $+9$ " -12 " $=0$ or $24A+36B-12C=0$
or if $A=1$, $B=C=-1$,
and there results the portion $a_2a_4-a_3^2-a_2^3$ of the complete form

$$a_2a_4-a_3^2-a_1^2a_4+2a_1a_2a_3-a_2^3;$$

the meaning is that we so determine A, B, C, that

$$\begin{aligned}&A(a_2-a_1^2)(a_4-4a_1a_3+6a_1^2a_2-3a_1^4)\\ &+B(a_3-3a_1a_2+2a_1^3)^2\\ &+C(a_2-a_1^2)^3,\end{aligned}$$

may be without the terms a_1^6, $a_1^2a_2^2$.

SEC. 4. *General form of a Seminvariant.*

For the quantic of order ρ, a seminvariant is of the form

$$u = \alpha\rho^{\kappa} + \beta\rho^{\kappa-1} + \gamma\rho^{\kappa-2} + \delta\rho^{\kappa-3} + \ldots + J,$$

(ρ being the letter of weight ρ) wherein $\alpha, \beta, \gamma, \delta, \ldots J$ are functions of the literal coefficients which do not involve ρ; forming the successive derivatives with regard to ρ, which are of course seminvariants, we have

$$\frac{du}{d\rho} = \kappa\alpha\rho^{\kappa-1} + (\kappa-1)\beta\rho^{\kappa-2} + (\kappa-2)\gamma\rho^{\kappa-3} + (\kappa-3)\delta\rho^{\kappa-4} + \ldots$$

$$\frac{d^2u}{d\rho^2} = \kappa(\kappa-1)\alpha\rho^{\kappa-2} + (\kappa-1)(\kappa-2)\beta\rho^{\kappa-3} + (\kappa-2)(\kappa-3)\gamma\rho^{\kappa-4} + (\kappa-3)(\kappa-4)\delta\rho^{\kappa-5} + \ldots$$

$$\vdots$$

$$\frac{d^{\kappa-1}u}{d\rho^{\kappa-1}} = \frac{\kappa!}{1!}\alpha\rho + (\kappa-1)!\,\beta$$

$$\frac{d^{\kappa}u}{d\rho^{\kappa}} = \kappa!\,\alpha.$$

From these κ equations are deduced

$$\kappa!\,\alpha = \frac{d^{\kappa}u}{d\rho^{\kappa}}$$

$$(\kappa-1)!\,\beta = \frac{d^{\kappa-1}u}{d\rho^{\kappa-1}} - \frac{d^{\kappa}u}{d\rho^{\kappa}}\rho$$

$$(\kappa-2)!\,\gamma = \frac{d^{\kappa-2}u}{d\rho^{\kappa-2}} - \frac{d^{\kappa-1}u}{d\rho^{\kappa-1}}\rho + \frac{1}{2!}\frac{d^{\kappa}u}{d\rho^{\kappa}}\rho^2$$

and generally

$$(\kappa-t+1)!\,\alpha_t = \frac{d^{\kappa-t+1}u}{d\rho^{\kappa-t+1}} - \frac{d^{\kappa-t+2}u}{d\rho^{\kappa-t+2}}\rho + \frac{1}{2!}\frac{d^{\kappa-t+3}u}{d\rho^{\kappa-t+3}}\rho^2 - \ldots + (-)^{t-1}\frac{1}{(t-1)!}\frac{d^{\kappa}u}{d\rho^{\kappa}}\rho^{t-1}.$$

Substituting these values in the expression for u there results on reduction,

$$(-)^{\kappa+1}\kappa!\,u = \frac{d^{\kappa}u}{d\rho^{\kappa}}\rho^{\kappa} - \kappa\frac{d^{\kappa-1}u}{d\rho^{\kappa-1}}\rho^{\kappa-1} + \kappa(\kappa-1)\frac{d^{\kappa-2}u}{d\rho^{\kappa-2}}\rho^{\kappa-2} - \ldots + (-)^{s}\frac{\kappa!}{(\kappa-s)!}\frac{d^{\kappa-s}u}{d\rho^{\kappa-s}}\rho^{\kappa-s} + \ldots + (-)^{\kappa+1}\kappa!\,J,$$

in which $s \not> \kappa - 1$.

In this form every seminvariant to the ρ^{ic} may be expressed.

The result may also be written as follows:

$$\left(\frac{d^{\kappa}u}{d\rho^{\kappa}},\ -1!\frac{d^{\kappa-1}u}{d\rho^{\kappa-1}},\ +2!\frac{d^{\kappa-2}u}{d\rho^{\kappa-2}},\ \ldots (-)^{\kappa}\kappa!\,u\between \rho, 1\right)^{\kappa} + (-)^{\kappa+1}\kappa!\,J = 0,$$

J being that part of u not involving ρ.

ROYAL MILITARY ACADEMY, Woolwich, England, June 16, 1885.